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Almost Linear Upper Bounds
on the Length of General
Davenport-Schinzel Sequences

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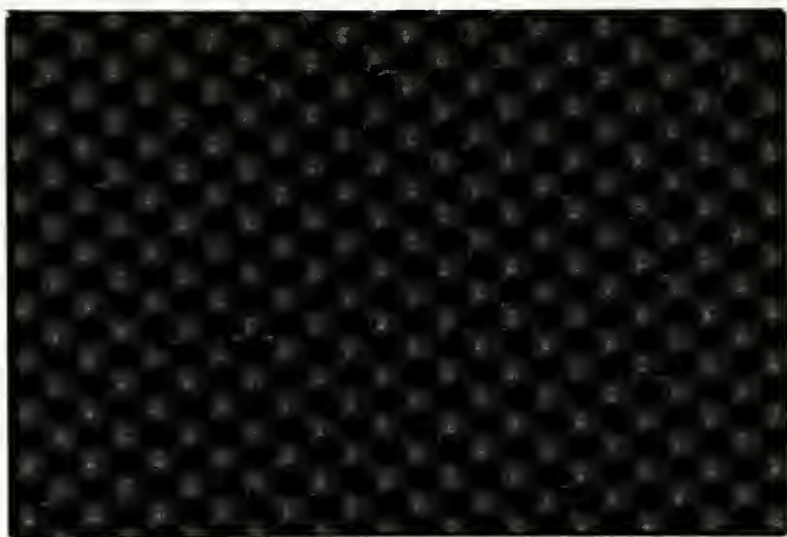Technical Report No. 199
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Almost Linear Upper Bounds on the Length of General Davenport-Schinzel Sequences

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ABSTRACT

Davenport-Schinzel sequences are sequences that do not contain forbidden subsequences of alternating symbols. They arise in the computation of the envelope of a set of functions. We obtain almost linear upper bounds on the length $\lambda_s(n)$ of Davenport-Schinzel sequences composed of n symbols in which no alternating subsequence is of length greater than $s + 1$. These bounds are of the form $O(n\alpha(n)^{O(\alpha(n)^{s-3})})$, and they generalize and extend the tight bound $\Theta(n\alpha(n))$ obtained by Hart and Sharir for the special case $s = 3$ ($\alpha(n)$ is the functional inverse of Ackermann's function), and also improve the upper bound $O(n \log^* n)$ due to Szemeredi.

1. Introduction

Consider the following combinatorial problem (most of this introductory material also appears in [HS]): Let n, s be positive integers. A sequence $U = (u_1, \ldots, u_m)$ of integers is an (n,s) *Davenport-Schinzel sequence* (a $DS(n,s)$ sequence for short), if it satisfies the following conditions:

(i) $1 \leq u_i \leq n$ for each i.

(ii) For each $i < m$ we have $u_i \neq u_{i+1}$.

(iii) There do not exist $s+2$ indices $1 \leq i_1 < i_2 < \cdots < i_{s+2} \leq m$ such that $u_{i_1} = u_{i_3} = u_{i_5} = \cdots = a$, $u_{i_2} = u_{i_4} = u_{i_6} = \cdots = b$, and $a \neq b$.

We will write $|U| = m$ for the *length* of the sequence U.

Define

$$\lambda_s(n) = \max \{|U| : U \text{ is a } DS(n,s) \text{ sequence} \} .$$

The goal of this paper is to establish almost linear upper bounds on $\lambda_s(n)$.

The problem of estimating $\lambda_s(n)$ has originally been posed by Davenport and Schinzel [DS]. Their interest in it arose from its connection to the

analysis of solutions of linear differential equations. Recently, Atallah [At] has raised it again independently, because of its significance for problems in dynamic computational geometry. These two applications are quite similar, and can be briefly described as follows. Let $f_1, \ldots, f_n$ be n real-valued continuous functions defined on a common interval I. Suppose that for each $i \neq j$ the functions f_i and f_j intersect in at most s points (e.g., this is the case for polynomials of fixed degree, or Chebycheff systems, and so on). Let $g(x) = \min \{f_i(x) : i=1, \ldots, n\}$, for $x \in I$, be the *lower envelope* (i.e. the pointwise minimum) of the f_i's, and let m be the smallest number of subintervals $I_1, \ldots, I_m$ of I such that for each k there exists an index i_k with $g(x) = f_{i_k}(x)$ for all $x \in I_k$. In other words, m is the number of connected portions of the graphs of the f_i's which constitute the graph of g. Assuming that $I_1, \ldots, I_m$ are arranged in this order from left to right, put

$$U(f_1, \ldots, f_n) = (i_1, \ldots, i_m) .$$

It is now easily seen that $U(f_1, \ldots, f_n)$ is a $DS(n,s)$ sequence. Moreover, it is known (see [At]) that for any $DS(n,s)$ sequence U one can construct a collection $f_1, \ldots, f_n$ of such functions for which $U(f_1, \ldots, f_n) = U$. Therefore the largest possible value of m is precisely $\lambda_s(n)$.

Thus, in this setting, Davenport-Schinzel sequences are strongly related to the problem of computing the (lower) envelope of a set of functions which intersect each other in pairs in at most some fixed number of points. This problem has many applications in computational geometry and related areas, many of which are given in [At] and in [HS].

The problem of estimating $\lambda_s(n)$ has been studied in several papers [DS], [Da], [RS], [Sz], [At], [HS]. It is known (and easy to prove) that $\lambda_1(n) = n$ and $\lambda_2(n) = 2n-1$. Hart and Sharir [HS] have shown that $\lambda_3(n) = \Theta(n\alpha(n))$ by establishing an equivalence between $DS(n,3)$ sequences and certain sequences of generalized path compressions on trees, and then by analyzing these path compression schemes. Here $\alpha(n)$ is a functional inverse of Ackermann's function, and is very slowly growing (see [HS] for more detail concerning Ackermann's function; a brief review of the properties of this function is also given in Section 3). For $s > 3$ the deep result of Szemeredi [Sz] states that $\lambda_s(n) = O(n \log^* n)$, where[#] the constant of proportionality depends on s, and where $\log^* n$ is the smallest $i \geq 1$ such that $\epsilon_i \geq n$, where $\epsilon_1 = 2$ and $\epsilon_{i+1} = 2^{\epsilon_i}$. (We use here the following standard notations. Let f, g be two positive functions defined over the integers. Then $f(n) = O(g(n))$ (resp. $f(n) = \Omega(g(n))$) if there exists a constant $C > 0$ such that $f(n) \leq Cg(n)$ (resp. $f(n) \geq Cg(n)$) for all n. We write $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold, namely, if there exist constants $C_1, C_2 > 0$ such that $C_1 g(n) \leq f(n) \leq C_2 g(n)$ for all n.)

In this paper we derive the following upper bounds on $\lambda_s(n)$ for arbitrary $s > 3$

[#] In this paper all logarithms are with base 2.

$$\lambda_s(n) \leq nQ_s(\alpha(n)) \tag{1.1}$$

where $Q_s(\alpha) = O(\alpha^{O(\alpha^{s-3})})$ (a more precise definition of Q_s is given in Section 5 below). These bounds improve Szemerédi's upper bounds [Sz]. We do not know however whether these bounds are tight (in fact we strongly suspect they are not). The proof of (1.1) involves a somewhat intricate inductive argument based on recurrence formulae which extend the formula used in [HS] for the case $s = 3$. Another feature of the proof given here is that we analyze Davenport Schinzel sequences directly, and do not use any reduction of them to other combinatorial structures, as was done in [HS]. The results of this paper can also be modified to yield a more direct proof of the upper bound on $\lambda_3(n)$ given in [HS].

2. Decomposition of Davenport-Schinzel sequences into chains

Definition: Let U be a $DS(n,s)$ sequence, and let $1 \leq t < s$. A *t-chain* c is a contiguous subsequence of U which is a Davenport-Schinzel sequence of order t.

Given n, s, t and U as above, we partition U into disjoint t-chains, proceeding from left to right in the following inductive manner. Suppose that the initial portion $(u_1, \ldots, u_j)$ of U has already been decomposed into t-chains. The next t-chain in our partitioning is then the largest subsequence of U of the form $(u_{j+1}, \ldots, u_k)$ which is still a Davenport-Schinzel sequence of order t. We refer to this partitioning as the *canonical decomposition* of U into t-chains, and let $m = m_t(U)$ denote the number of t-chains in this decomposition.

The problem of obtaining good upper bounds for the quantities

$$\mu_{s,t}(n) = \max \{ m_t(U) : U \text{ is a } DS(n,s) \text{ sequence } \}$$

seems quite hard for general s and t. Nevertheless we have the following partial results.

Proposition 2.1 $\mu_{s,s-1}(n) \leq n$ and $\mu_{s,s-2}(n) \leq 2n - 1$.

Proof: Consider the first inequality. Let U be a $DS(n,s)$ sequence, and let $(c_1, \ldots, c_m)$ be the canonical decomposition of U into $(s-1)$-chains. Let $1 \leq j < m$, and suppose that $c_j = (u_p, u_{p+1}, \ldots, u_q)$. Let $b = u_{q+1}$. By our construction c_j is a Davenport-Schinzel sequence of order $s-1$, whereas $c_j \, \| \, [b]$ is not. Hence there exists another symbol $a \neq b$ such that c_j contains a subsequence of s alternations of a and b ending at a. If s is odd then this subsequence has the form

$$a \quad b \quad a \quad \cdots \quad b \quad a \, .$$

But then c_j must contain both the first occurrence of b and the last occurrence of a in U, for otherwise U would plainly contain one of the forbidden subsequences $a \, b \, a \, \cdots \, b \, a$ or $b \, a \, b \, \cdots \, a \, b$ of length $s+2$. This implies that each chain c_j must contain the last occurrence of some symbol (this obviously holds also for $j = m$), and hence we must have $m \leq n$. Similarly, if s is even then c_j must contain the subsequence

$$b \quad a \quad b \quad \cdots \quad b \quad a$$

of length s, and similar arguments to those used above imply that c_j must contain both the first and the last occurrence of a, from which it follows once again that $m \leq n$.

To estimate $\mu_{s,s-2}(n)$ we proceed in much the same way. Specifically, suppose s is odd, and consider an $(s-2)$-chain c_j in the canonical decomposition of U, followed by a symbol b. Then there exists $a \neq b$ such that c_j contains a subsequence of the form

$$b \quad a \quad b \quad \cdots \quad b \quad a$$

of length $s-1$. Now c_j must contain either the first or the last occurrence of a in U, for if c_j contained neither of these occurrences, U would have to contain a forbidden alternating subsequence of the form

$$a \quad b \quad a \quad b \quad \cdots \quad b \quad a \quad b \quad a$$

of length $s+2$. Since there are n first occurrences and n last occurrences of symbols in U, and since c_1 plainly contains at least two such first occurrences, it follows that $\mu_{s,s-2}(n) \le 2n - 1$. Similar arguments can be used to handle the case in which s is even. Q.E.D.

Definition: Let n, m and s be positive integers. We denote by $\psi_s(m,n)$ the maximal length of a $DS(n,s)$ sequence composed of at most m 1-chains.

In Section 4 we will obtain upper bounds for $\psi_s(m,n)$. In order to apply these estimates to obtain upper bounds on $\lambda_s(n)$, we will need to estimate $\mu_{s,1}(n)$, a task which we have been unable to handle successfully (for $s > 3$). Instead, we will use the following approximation. Let U be a $DS(n,s)$ sequence of maximal length, and let $(c_1, \ldots, c_m)$, $m \le n$, be its canonical decomposition into $(s-1)$-chains. We will replace each $(s-1)$-chain c_j by a 1-chain c_j^*, by leaving in c_j only one occurrence of each symbol appearing there. We will show in Section 5 that in this way we can obtain a $DS(n,s)$ sequence U^* composed of at most n chains, whose length is not much smaller than the length of U, i.e. than $\lambda_s(n)$. This will establish a connection between $\lambda_s(n)$ and $\psi_s(n,n)$, from which the desired upper bound on $\lambda_s(n)$ will be obtained.

3. Statement of Main Results

In this section we state our main results, concerning the functions λ and ψ. For this purpose, we briefly review first the definition and some basic properties of Ackermann's function and certain auxiliary functions ("generalized exponentials" - cf. [Ac]). A more detailed description of these functions can be found in [HS].

Let N be the set of positive integers 1,2,... . Given a function g from a set into itself, denote by $g^{(s)}$ the composition $g \circ g \circ \cdots \circ g$ of g with itself s times, for $s \in N$. Define inductively a sequence $\{A_k\}_{k=1}^{\infty}$ of functions from N into itself as follows:

$$A_1(n) = 2n$$
$$A_k(n) = A_{k-1}^{(n)}(1) , \quad k \geq 2$$

for all $n \in N$. Note that for all $k \geq 2$, the function A_k satisfies

$$A_k(1) = 2 ,$$
$$A_k(n) = A_{k-1}(A_k(n-1)) , \quad n \geq 2 .$$

In particular, $A_2(n) = 2A_2(n-1)$, thus A_2 is the "power function"

$$A_2(n) = 2^n , \quad n \in N .$$

Then $A_3(n) = 2^{A_3(n-1)}$, thus A_3 is the "tower function"

$$A_3(n) = 2^{2^{\cdot^{\cdot^{2}}}} ,$$

with n 2's in the exponential tower, for $n \in N$. Finally, put

$$A(n) = A_n(n) .$$

This is *Ackermann's function* (this is the variant used in [HS]). Ackermann's function A grows very fast; its first values are: $A(1) = 2$, $A(2) = 4$, $A(3) = 16$ and $A(4)$ is a tower of 65536 2's.

Given a strictly increasing fuction g from N into itself, its *functional inverse* is the function γ from N into itself given by

$$\gamma(n) = \min \{s \geq 1 : g(s) \geq n\} ;$$

thus, $\gamma(n) = s$ if and only if $g(s-1) < n \leq g(s)$. In particular, let α_k and α denote the functional inverses of A_k and A, respectively. Then, for all $n \in N$,

$$\alpha_1(n) = \left\lceil \frac{n}{2} \right\rceil ,$$

$$\alpha_2(n) = \lceil \log n \rceil .$$

The functions α_k are easily seen to satisfy the following recursive formula:

$$\alpha_k(n) = \min \{s \geq 1 : \alpha_{k-1}^{(s)}(n) = 1\} ;$$

that is, $\alpha_k(n)$ is the number of iterations of α_{k-1} needed to go from n to 1. In particular, $\alpha_3(n)$ is precisely $\log^* n$, as defined in Section 1.

All the functions α_k are non-decreasing, and converge to infinity with their argument. The same holds for α too, which grows more slowly than any of the α_k. Note that $\alpha(n) \leq 4$ for all $n \leq A(4)$ which is a tower with 65536 2's, thus $\alpha(n) \leq 4$ for all practical purposes.

The following property, which follows immediately from the above definitions, will be used in the sequel

$$\alpha_{\alpha(n)}(n) = \alpha(n) . \tag{3.1}$$

We will also need the following inequality, proved in [HS]:

$$\alpha_{2\alpha(n)-3}(n) \leq 3 . \tag{3.2}$$

We can now state our results.

MAIN THEOREM:

$$\lambda_s(n) = O(n\alpha(n)^{O(\alpha(n)^{s-3})})$$

for $s \geq 3$ and $n \geq 1$.

It is easily verified that these bounds improve those of Szemerédi (which can be stated as $\lambda_s(n) \leq C(s)n\alpha_3(n))$, for each s and sufficiently large n.

The Main Theorem is proven in Section 5. It is a consequence of the following Theorem, which yields upper bounds on the length of Davenport-Schinzel sequences with a given number of 1-chains (i.e. on the function $\psi_s(m,n)$ defined in the preceding section).

Theorem A:

$$\psi_s(m,n) = O((m+n)\alpha(m)^{O(\alpha(m)^{s-3})}) .$$

Theorem A will be proved in the following section.

4. Upper bounds for $\psi_s(m,n)$

In this section we establish almost linear upper bounds on the maximal length $\psi_s(m,n)$ of a $DS(n,s)$ sequence composed of at most m 1-chains. Consider first the case $s = 3$. This case has been analyzed in [HS], and the following upper bounds have been obtained (actually the bounds derived in [HS] were for another function, namely the maximal length of admissible path compression schemes on trees, but it can be easily verified (cf. Proposition 4.1 of [HS]) that these same bounds also apply to our function ψ_s):

$$\psi_3(m,n) \leq 4km\alpha_k(m) + 2kn , \tag{4.1}$$

for each $k \geq 1$. These bounds also yield, putting $k = 2\alpha(m) - 3$ and making

use of (3.2),

$$\psi_3(m,n) \le (24m + 4n)\,\alpha(m)\,. \tag{4.2}$$

Our aim is to establish inductively similar inequalities of the form

$$\psi_s(m,n) \le mF_\omega^{(s)}(\alpha(m)) + nG_\omega^{(s)}(\alpha(m))\,, \tag{4.3}$$

where $F_\omega^{(s)}$, $G_\omega^{(s)}$ are appropriate positive and monotonic functions, whose precise form will be specified below. Clearly, for the initial case $s = 3$, (4.3) is obtained from (4.2) by choosing

$$F_\omega^{(3)}(\alpha) = 24\alpha\,, \qquad G_\omega^{(3)}(\alpha) = 4\alpha\,. \tag{4.4}$$

Suppose next that (4.3) has been established for $s-1$, and consider the case of s. Our aim is to establish a recurrence formula for $\psi_s(m,n)$ which is a generalization of the formula obtained in Proposition 5.1 of [HS]. Specifically, we have

Proposition 4.1 Let m, $n \ge 1$, and let $b > 1$ be a divisor of m. Then there exist integers n^*, n_1, n_2, . . . ,$n_b \ge 0$ such that

$$n^* + \sum_{i=1}^{b} n_i = n\,,$$

and

$$\psi_s(m,n) \le \sum_{i=1}^{b} \psi_s(\frac{m}{b},n_i) + 2\psi_s(b,n^*)\cdot G_\omega^{(s-1)}(\alpha(m)) + \tag{4.5}$$
$$m\left(3 + 2F_\omega^{(s-1)}(\alpha(m)) + 2G_\omega^{(s-1)}(\alpha(m))\right)\,.$$

Proof: Let U be a $DS(n,s)$-sequence consisting of at most m 1-chains $c_1, . . . ,c_m$ such that $|U| = \psi_s(m,n)$, and let $b > 1$ be a divisor of m. Partition the sequence U into b layers $L_1, . . . ,L_b$ so that the layer L_i consists of the $p = \frac{m}{b}$ 1-chains $c_{(i-1)p+1}$, $c_{(i-1)p+2}$, . . . ,c_{ip}. Call a symbol a *internal*

to layer L_l if all the occurrences of a in U are within L_l. A symbol will be called *external* if it is not internal to any layer. Suppose that there are n_l internal symbols in layer L_l and n^* external symbols (thus $n^* + \sum_{l=1}^{b} n_l = n$).

To estimate the total number of occurrences in U of symbols that are internal to L_l, we proceed as follows. Erase from L_l all external symbols. Next scan L_l from left to right and erase each element which has become equal to the element immediately preceding it. This leaves us with a sequence L_l^* which is clearly a $DS(n_l,s)$ sequence consisting of at most $\frac{m}{b}$ 1-chains, and thus its length is at most $\psi_s(\frac{m}{b},n_l)$. Moreover, if two equal internal elements in L_l have become adjacent after erasing the external symbols, then these two elements must have belonged to two distinct 1-chains, thus the total number of deletions of internal symbols is at most $\frac{m}{b}$.

Hence, summing over all layers, we conclude that the total contribution of internal symbols to $|U|$ is at most

$$ m + \sum_{l=1}^{b} \psi_s(\frac{m}{b},n_l) . $$

Next, to estimate the contribution of external symbols to $|U|$, we argue as follows. For each L_l consider separately external symbols whose first occurrence in U appear to the left of L_l (call them *non-starting* external symbols), and all the other external symbols which appear in L_l. Note that these latter symbols cannot have their rightmost occurrence within L_i, and for that reason we will refer to them as *non-ending* external symbols. Consider

first non-starting symbols. To estimate their contribution to the length of L_l we erase from L_l all other symbols occuring there, and, if necessary, also erase each occurrence of a non-starting symbol which has become equal to the element immediately preceding it. As above, at most $\frac{m}{b} - 1$ deletions of external non-starting symbols will be performed. Let L_l^* be the resulting subsequence, and suppose that it is composed of p_l distinct symbols.

We claim that L_l^* is a $DS(p_l, s-1)$ sequence. Indeed, if this were not the case, L_l^* would contain an alternating subsequence W of two external non-starting symbols a, b, which has length $s+1$ and which begins, say, with a. But since b is assumed to be non-starting, there must exist at least one occurrence of b to the left of L_l, which, concatenated with W, yields a forbidden alternating subsequence of length $s+2$ within U, contrary to assumptions.

Thus L_l^* is a $DS(p_l, s-1)$ sequence, consisting of at most $\frac{m}{b}$ 1-chains, so that, by induction hypothesis, its length is at most

$$\psi_{s-1}(\frac{m}{b}, p_l) \leq \frac{m}{b} F_\omega^{(s-1)}(\alpha(\frac{m}{b})) + p_l G_\omega^{(s-1)}(\alpha(\frac{m}{b})) .$$

Hence, summing over all layers, the total contribution of external non-starting symbols is at most

$$m - b + m F_\omega^{(s-1)}(\alpha(m)) + \left[\sum_{j=1}^{b} p_i\right] G_\omega^{(s-1)}(\alpha(m)) .$$

But if we leave, in each L_l, the leftmost occurrence of each external non-starting symbol, and furthermore delete (at most b) elements which have become equal to those immediately preceding them, we obtain a $DS(n^*, s)$

sequence composed of at most b chains, whose total length is at least $\sum_{l=1}^{b} p_l - b$. Thus

$$\sum_{l=1}^{b} p_l \leq b + \psi_s(b,n^*) .$$

Hence the total contribution of the external non-starting symbols to $|U|$ is at most

$$m - b + mF_\omega^{(s-1)}(\alpha(m)) + \left(b + \psi_s(b,n^*)\right)G_\omega^{(s-1)}(\alpha(m)) .$$

Repeating the above argument once again to the analysis of the number of occurrences of external non-ending symbols in each of the layers L_l, we conclude that total contribution of these symbols to $|U|$ is also bounded by the bound just stated. Thus we obtain the following final inequality

$$\psi_s(m,n) \leq \sum_{l=1}^{b} \psi_s(\frac{m}{b},n_l) + 3m - 2b + 2mF_\omega^{(s-1)}(\alpha(m)) +$$
$$2\left(b + \psi_s(b,n^*)\right)G_\omega^{(s-1)}(\alpha(m)) \leq$$
$$\sum_{l=1}^{b} \psi_s(\frac{m}{b},n_l) + 3m + 2m\left(F_\omega^{(s-1)}(\alpha(m)) + G_\omega^{(s-1)}(\alpha(m))\right) +$$
$$2\psi_s(b,n^*) G_\omega^{(s-1)}(\alpha(m)) ,$$

as asserted. Q.E.D.

Proposition 4.2: Let $n, q \geq 1$, $k \geq 2$, and suppose that m divides $A_k(q)$. Then

$$\psi_s(m,n) \leq mqF_k^{(s)}(\alpha(A_k(q))) + nG_k^{(s)}(\alpha(A_k(q))) , \tag{4.6}$$

where

$$F_1^{(s)}(\alpha) = 0 , \qquad G_1^{(s)}(\alpha) = 2 ,$$

and

$$F_k^{(s)}(\alpha) = 2F_{k-1}^{(s)}(\alpha) \cdot G_\omega^{(s-1)}(\alpha) + 3 + 2F_\omega^{(s-1)}(\alpha) + 2G_\omega^{(s-1)}(\alpha) \tag{4.7}$$

$$G_k^{(s)}(\alpha) = 2G_{k-1}^{(s)}(\alpha) \cdot G_\omega^{(s-1)}(\alpha) ,$$

for $k \geq 2$.

Proof: We will use (4.5) repeatedly to obtain the series of upper bounds on ψ_s, stated in (4.6) for $k=2,3,\ldots$. At each step we choose b in an appropriate manner, and estimate $\psi_s(b,n^*)$ using the bound obtained in the preceding step. This yields a recurrence relation on ψ_s which we solve to obtain a better upper bound on ψ_s.

Specifically, we proceed by double induction on k and q. To start this iterative process with $k = 2$, suppose first that $m = A_2(q) = 2^q$. Choose $b = 2$ in (4.5); it is easily checked that $\psi_s(2,n^*) = 2n^*$ for all n^*, so that (4.5) yields

$$\psi_s(m,n) \leq m\left(3 + 2F_\omega^{(s-1)}(\alpha(m)) + 2G_\omega^{(s-1)}(\alpha(m))\right) +$$
$$4n^* \, G_\omega^{(s-1)}(\alpha(m)) + \psi_s(\tfrac{m}{2},n_1) + \psi_s(\tfrac{m}{2},n_2) .$$

The solution to this recurrence relation, for m a power of 2 and $n = n^* + n_1 + n_2$ arbitrary, is

$$\psi_s(m,n) \leq m \log m \cdot \left(3 + 2F_\omega^{(s-1)}(\alpha(m)) + 2G_\omega^{(s-1)}(\alpha(m))\right) + 4nG_\omega^{(s-1)}(\alpha(m)) .$$

(This is easily established by induction on m.)

To complete the argument for $k = 2$, note that if m divides $A_2(q) = 2^q$ then m is a power of 2 and $\log m \leq q$, thus

$$\psi_s(m,n) \leq mq \cdot \left(3 + 2F_\omega^{(s-1)}(\alpha(m)) + 2G_\omega^{(s-1)}(\alpha(m))\right) + 4nG_\omega^{(s-1)}(\alpha(m)) .$$

Thus if we put, as in (4.7),

$$F_2^{(s)}(\alpha) = 3 + 2F_\omega^{(s-1)}(\alpha) + 2G_\omega^{(s-1)}(\alpha) ,$$

and

$$G_2^{\{s\}}(\alpha) = 4G_\omega^{(s-1)}(\alpha) ,$$

we can rewrite the preceding inequality as

$$\psi_s(m,n) \le mqF_2^{\{s\}}(\alpha(m)) + nG_2^{\{s\}}(\alpha(m)) ,$$

which, since $m \le A_2(q)$, implies (4.6) for $k = 2$.

In particular, we have for $m = 1, 2$ (and $q = 1$)

$$\psi_s(m,n) \le mF_2^{\{s\}}(\alpha(m)) + nG_2^{\{s\}}(\alpha(m)) \le mF_k^{\{s\}}(\alpha(m)) + nG_k^{\{s\}}(\alpha(m)) ,$$

for all $k \ge 2$. Since $A_k(1) = 2$ it follows that (4.6) holds for each $k \ge 2$ and $q = 1$.

Suppose next that $k > 2$ and $q > 1$, and that the induction hypothesis is true for all $k' < k$ and $q' \ge 1$, and for $k' = k$ and all $q' < q$. Observe that $A_k(q-1)$ is a divisor of $A_k(q)$ because they are both powers of 2. Assume first that $m = A_k(q)$; let $t = A_k(q-1)$ and choose $b = \dfrac{m}{t}$, which is an integer dividing $m = A_k(q) = A_{k-1}(t)$. Hence by induction hypothesis (for $k-1$ and t) we have

$$\psi_s(b,n^*) \le \frac{m}{t} \, tF_{k-1}^{\{s\}}(\alpha(A_{k-1}(t))) + n^*G_{k-1}^{\{s\}}(\alpha(A_{k-1}(t))) =$$
$$mF_{k-1}^{\{s\}}(\alpha(m)) + n^*G_{k-1}^{\{s\}}(\alpha(m)) .$$

Then (4.5) becomes

$$\psi_s(m,n) \le \sum_{i=1}^{b} \psi_s(t,n_i) + 3m + 2m\left(F_\omega^{(s-1)}(\alpha(m)) + G_\omega^{(s-1)}(\alpha(m))\right) +$$
$$\left(2mF_{k-1}^{\{s\}}(\alpha(m)) + 2n^*G_{k-1}^{\{s\}}(\alpha(m))\right) \cdot G_\omega^{(s-1)}(\alpha(m)) ,$$

Recall that we have defined

$$F_k^{\{s\}}(\alpha) = 2F_{k-1}^{\{s\}}(\alpha) \cdot G_\omega^{(s-1)}(\alpha) + 3 + 2F_\omega^{(s-1)}(\alpha) + 2G_\omega^{(s-1)}(\alpha) ,$$
$$G_k^{\{s\}}(\alpha) = 2G_{k-1}^{\{s\}}(\alpha) \cdot G_\omega^{(s-1)}(\alpha) ,$$

Hence, using the induction hypothesis once more (for k and $q-1$), we obtain

$$\psi_s(m,n) \leq \sum_{l=1}^{b} \left[t(q-1)F_k^{\{s\}}(\alpha(t)) + n_l G_k^{\{s\}}(\alpha(t)) \right] +$$
$$mF_k^{\{s\}}(\alpha(m)) + n^*G_k^{\{s\}}(\alpha(m)) \leq$$
$$mqF_k^{\{s\}}(\alpha(m)) + nG_k^{\{s\}}(\alpha(m))$$

(because $n^* + \sum_{l=1}^{b} n_l = n$), as asserted.

Finally, assume m divides $A_k(q)$, say $A_k(q) = pm$. Let U be a $DS(n,s)$ sequence composed of at most m 1-chains whose length is $\psi_s(m,n)$. Create p copies of U, using p disjoint sets of n symbols each for these copies, and then concatenate all these copies to form a new sequence U^*. U^* is obviously a $DS(pn,s)$ sequence composed of at most $pm = A_k(q)$ 1-chains, and its length is therefore

$$p\psi_s(m,n) \leq \psi_s(A_k(q),pn) \leq qA_k(q)F_k^{\{s\}}(\alpha(A_k(q))) + pnG_k^{\{s\}}(\alpha(A_k(q))) ,$$

which, divided by p, yields the required inequality. This completes the proof. Q.E.D.

Corollary 4.3: For all m, $n \geq 1$ and $k \geq 2$,

$$\psi_s(m,n) \leq 2m\alpha_k(m)F_k^{\{s\}}(\alpha(A_k(\alpha_k(m)))) + nG_k^{\{s\}}(\alpha(A_k(\alpha_k(m)))) ,$$

where α_k is the functional inverse of A_k as defined in Section 3.

Proof: Put $q = \alpha_k(m)$, so that $A_k(q-1) < m \leq A_k(q)$. Let $p = \left\lceil \dfrac{A_k(q)}{m} \right\rceil \geq 1$, then $A_k(q) < (p+1)m \leq 2pm$. As in the preceding proof we have

$$p\psi_s(m,n) \leq \psi_s(pm,pn) \leq \psi_s(A_k(q),pn) \leq$$
$$qA_k(q)F_k^{\{s\}}(\alpha(A_k(q))) + pnG_k^{\{s\}}(\alpha(A_k(q))) \leq$$
$$2pmqF_k^{\{s\}}(\alpha(A_k(q))) + pnG_k^{\{s\}}(\alpha(A_k(q))) .$$

Dividing by p we obtain the desired inequality. Q.E.D.

Corollary 4.4: For all $m, n \geq 1$,

$$\psi_s(m,n) \leq m F_\omega^{(s)}(\alpha(m)) + n G_\omega^{(s)}(\alpha(m)) ,$$

where

$$F_\omega^{(s)}(\alpha) = 2\alpha F_\alpha^{(s)}(\alpha) , \qquad G_\omega^{(s)}(\alpha) = G_\alpha^{(s)}(\alpha) .$$

Proof: Put $k = \alpha(m)$. Then (3.1) implies that $\alpha_k(m) = \alpha(m)$, so that

$$\alpha(A_k(\alpha_k(m))) = \alpha(A(\alpha(m))) = \alpha(m) .$$

Hence Corollary 3.1 implies that

$$\psi_s(m,n) \leq 2m\alpha(m) F_{\alpha(m)}^{(s)}(\alpha(m)) + n G_{\alpha(m)}^{(s)}(\alpha(m)) ,$$

as asserted. Q.E.D.

Proposition 4.2 and Corollary 4.4 thus complete the inductive proof of (4.3) for all $s \geq 3$.

Proof of Theorem A: The inequalities (4.4) and (4.7) together yield a recurrence scheme for the functions $F_\omega^{(s)}$ and $G_\omega^{(s)}$. We can use these formulae to obtain more direct estimates on these functions as follows.

Proposition 4.5: For each $s > 3$,

$$G_\omega^{(s)}(\alpha) \leq 8^{\alpha^{s-3}} \alpha^{(\alpha-1)^{s-3}} , \qquad (4.8a)$$
$$F_\omega^{(s)}(\alpha) \leq 11(2\alpha)^{s-3} G_\omega^{(s)}(\alpha) . \qquad (4.8b)$$

Proof: We will first establish (4.8a) using induction on s. For $s = 3$ (4.8a) follows immediately from (4.4). Suppose $s > 3$ and that (4.8a) has been established for $s-1$. It easily follows from (4.7) that

$$G_\omega^{(s)}(\alpha) \leq 2\left[2G_\omega^{(s-1)}(\alpha)\right]^{\alpha-1} .$$

Thus by induction hypothesis we have

$$G_\omega^{(s)}(\alpha) \leq 2\left[2 \cdot 8^{\alpha^{s-4}} \alpha^{(\alpha-1)^{s-4}}\right]^{\alpha-1}$$

$$\leq 8^{(\alpha^{s-4} + 1)(\alpha - 1) + 1} \cdot_\alpha (\alpha - 1)^{s-3}$$
$$\leq 8^{\alpha^{s-3}} \alpha (\alpha - 1)^{s-3} ,$$

as asserted. This proves (4.8a) for all $s \geq 3$. Next consider (4.8b). It follows immediately from (4.4) that (4.8b) holds for $s = 3$. Suppose $s > 3$ and that (4.8b) has been established for $s-1$. Put

$$a = 2G_\omega^{(s-1)}(\alpha) , \qquad b = 3 + 2F_\omega^{(s-1)}(\alpha) + 2G_\omega^{(s-1)}(\alpha) .$$

Then (4.7) implies

$$F_k^{(s)}(\alpha) \leq a F_{k-1}^{(s)}(\alpha) + b$$

where $F_1^{(s)}(\alpha) = 0$. This implies that

$$F_k^{(s)}(\alpha) \leq \frac{a^{k-1} - 1}{a - 1} b .$$

Thus, putting $k = \alpha$, it follows that

$$F_\omega^{(s)}(\alpha) \leq \frac{G_\omega^{(s)}(\alpha) - 2}{2G_\omega^{(s-1)}(\alpha) - 1} \cdot \left(3 + 2F_\omega^{(s-1)}(\alpha) + 2G_\omega^{(s-1)}(\alpha) \right) \cdot \alpha .$$

To simplify the calculations, write (4.8b) in the form

$$F_\omega^{(s)}(\alpha) \leq D_s(\alpha) G_\omega^{(s)}(\alpha) . \tag{4.9}$$

Then the induction hypothesis implies that

$$F_\omega^{(s)}(\alpha) \leq \frac{\left[3 + 2(D_{s-1}(\alpha) + 1)G_\omega^{(s-1)}(\alpha) \right] \cdot \alpha}{2G_\omega^{(s-1)}(\alpha) - 1} \cdot G_\omega^{(s)}(\alpha)$$
$$\leq (2D_{s-1}(\alpha) + 5) \alpha G_\omega^{(s)}(\alpha) .$$

Hence (4.9) will continue to hold for s too if we put

$$D_s(\alpha) = (2D_{s-1}(\alpha) + 5) \alpha .$$

The solution of this last recurrence formula is

$$D_s(\alpha) = (2\alpha)^{s-3} \cdot D_3(\alpha) + \frac{(2\alpha)^{s-3} - 1}{2\alpha - 1} \cdot 5\alpha$$
$$\leq (2\alpha)^{s-3} \cdot D_3(\alpha) + 2 \cdot (2\alpha)^{s-4} \cdot 5\alpha$$
$$= (2\alpha)^{s-3} \cdot (D_3(\alpha) + 5) = 11(2\alpha)^{s-3} ,$$

from which (4.8b) follows immediately. Q.E.D.

Theorem A now follows immediately from (4.3) and (4.8).

Corollary 4.6:

$$\psi_s(n,n) \leq 12n(2\alpha(n))^{s-3} \cdot 8^{\alpha(n)^{s-3}} \cdot \alpha(n)^{(\alpha(n)-1)^{s-3}}.$$

Proof: Immediate.

5. Upper bounds for $\lambda_s(n)$

Having obtained the upper bounds on $\psi_s(m,n)$ in the preceding section, our next step is to use them to obtain almost linear upper bounds for $\lambda_s(n)$, by establishing a relationship between these two quantities. As already noted in Section 2, this will be achieved by constructing a $DS(n,s)$ sequence U^* whose length is sufficiently close to $\lambda_s(n)$, but which contains only at most n 1-chains, and then by using U^* to bound $\lambda_s(n)$ in terms of $\psi_s(n,n)$.

Details are as follows. We will prove, using induction on s, that for each $n \geq 1$

$$\lambda_s(n) \leq nQ_s(\alpha(n)) \tag{5.1}$$

where Q_s is a positive and monotonic function whose exact form will be specified below. The claim holds for $s = 2$ if we put $Q_2(\alpha) = 2$, as follows from the results of [At] and [HS].

Suppose the claim holds for all $2 \leq s' < s$ and consider $\lambda_s(n)$. Let U be a $DS(n,s)$ sequence of maximal length $\lambda_s(n)$, and let $(c_1,c_2,\ldots,c_m)$ be the canonical decomposition of U into $(s-1)$-chains. By Proposition 2.1 we have $m \leq n$. For each $j = 1,\ldots,m$ replace c_j by the 1-chain c_j^* obtained by retaining in c_j only the leftmost occurrence of each symbol appearing there.

To obtain U^* we first concatenate the 1-chains $c_1^*, \ldots, c_m^*$ in order; then, proceeding from left to right, we erase every element that has become equal to the element immediately preceding it (at most $m \leq n$ elements will be erased). Suppose that c_j contains k_j distinct symbols, for $j = 1, \ldots, m$. Since each c_j is a $DS(k_j, s-1)$-sequence, we have by induction hypothesis

$$|c_j| \leq \lambda_{s-1}(k_j) \leq k_j Q_{s-1}(\alpha(k_j)) \leq k_j Q_{s-1}(\alpha(n)) .$$

Thus

$$|c_j^*| = k_j \geq \frac{|c_j|}{Q_{s-1}(\alpha(n))}$$

so that

$$|U^*| \geq \sum_{j=1}^{m} |c_j^*| - n \geq \frac{\sum\limits_{j=1}^{m} |c_j|}{Q_{s-1}(\alpha(n))} - n .$$

Hence

$$\lambda_s(n) = |U| = \sum_{j=1}^{m} |c_j| \leq (n + |U^*|) \cdot Q_{s-1}(\alpha(n)) .$$

But $|U^*|$ is plainly a $DS(n,s)$ sequence composed of at most n 1-chains, so that by Theorem A

$$|U^*| \leq \psi_s(n,n) \leq nH_s(\alpha(n)) ,$$

where $H_s(\alpha(n))$ is as given in Corollary 4.6. We thus conclude that

$$\lambda_s(n) \leq n (H_s(\alpha(n)) + 1)Q_{s-1}(\alpha(n)) .$$

Hence (5.1) will also hold for s if we put

$$Q_s(\alpha) = (H_s(\alpha) + 1) \cdot Q_{s-1}(\alpha) .$$

If we substitute the explicit form of H_s into this recurrence formula and solve it, we obtain

$$Q_s(\alpha) \leq 2 \cdot 13^{s-2}(2\alpha)^{\frac{(s-3)(s-2)}{2}} 8^{\alpha^0 + \alpha^1 + \cdots + \alpha^{s-3}} \alpha^{(\alpha-1)^0 + (\alpha-1)^1 + \cdots + (\alpha-1)^{s-3}}$$

This completes the proof of the Main Theorem.

Remarks: (1) The bounds just obtained could be somewhat improved if we used the canonical decomposition of U into $(s-2)$-chains rather than into $(s-1)$-chains. Nevertheless, the resulting bounds would be of the same asymptotic order of magnitude.

(2) The case $s = 3$ can also be handled by the above technique. In this case the chains c_j are 2-chains, so that the length of each c_j is at most $2k_j - 1$ (cf. [At], [HS]). The same reasoning as above would then imply

$$\lambda_3(n) \leq 2(n + \psi_3(n,n)) .$$

If we use this bound, and somewhat improve the analysis in Section 4 for the case $s = 3$, we can obtain a more direct derivation of the bound $\lambda_3(n) = O(n\alpha(n))$ obtained in [HS].

(3) We do not have lower bounds for $\lambda_s(n)$ (other than those obtained in [HS] for λ_3), and we strongly suspect that the bounds (5.1) can be substantially improved. To do that, using the techniques developed in this paper, one would have (a) to improve the formula (4.5) by finding tighter estimates on the contribution of external symbols to the length of U, and (b) to establish a better connection between λ_s and ψ_s, e.g. by establishing an explicit upper bound on $\mu_{s,1}(n)$.

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